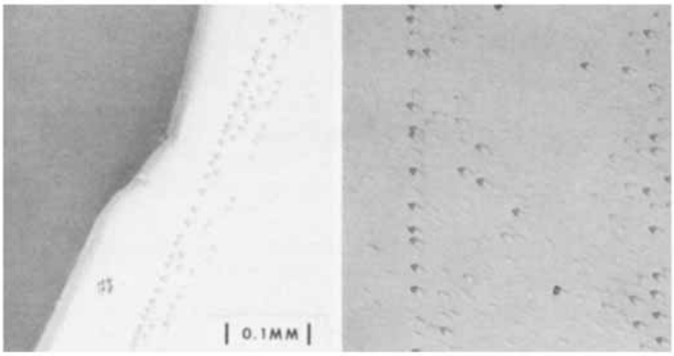

Fig. 13. Stacking faults at edge and center of slice half immersed in water in a polypropylene vessel between etching and deposition.

Table 1 summarizes the cleaning steps tested and the results. Figure 12 shows the appearance of the slices after the deposition test. The main results noted were that the polypropylene vessel used for water and HF rinsing gave off a contaminant film, and that the peroxide was also heavily loaded with a contaminant that presented problems for the 998°C etching and deposition. Figure 13 shows two regions, an edge and the center, of the slice partially immersed in water in a polypropylene vessel before deposition. Stacking faults were arranged in rows in each of these regions, typical of contamination having been left from a film on the water. Replacing the polypropylene vessel with Teflon* eliminated this film.

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Controller Design for Distributed Systems via Bass' Technique

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Bass' feedback controller design technique is extended to distributed parameter systems and its use is illustrated for a parabolic system with boundary control.

The development of the design technique rests on a conjecture concerning the necessary and sufficient conditions for the asymptotic stability of linear time invariant partial differential equations. It is shown through a study of the discretized analog of a distributed system that the most likely candidate for a Lyapunov functional general enough to yield the necessary and sufficient conditions for asymptotic stability is a double integral with a symmetric kernel.

The essence of Bass' design technique (*1*) is that it aims to maximize stability and optimize system performance. The control variables are chosen to make the rate of change of a quadratic Lyapunov function as negative as possible, while the parameters of the Lyapunov function are chosen so that the control nearly minimizes a specified quadratic performance index. For linear lumped parameter processes with hard constraints on the control, the design technique proceeds as follows:

Let process equations be given by

$$\frac{dy}{dt} = Ay + Bm \tag{1}$$

where y = an n dimensional vector
m = an m dimensional vector whose elements satisfy $|m_i| \leq k_i$
A = an $n \times n$ real matrix all of whose eigenvalues have negative real parts
B = an $n \times m$ real matrix

The performance criterion to be minimized is $\phi(m)$, where

$$\phi(m) = \int_0^\infty y'Qydt \tag{2}$$

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Q = an $n \times n$ positive definite matrix (in some special cases Q may be semidefinite). We take as our candidate for a Lyapunov function $V(t)$,

$$V(t) = y'Py \tag{3}$$

where P is an $n \times n$ positive definite matrix. Then

$$\dot{V}(t) = y'(A'P + PA)y + 2y'PBm \tag{4}$$

and $\dot{V}(t)$ is minimized by taking

$$m_i = -k_i \text{ sign } (B'Py)_i; \qquad (B'Py)_i = i\text{th element of } B'Py \tag{5}$$

Also, if we can choose the matrix P to satisfy

$$A'P + PA = -Q \tag{6}$$

then the control given by Equation (5) minimizes the integrand of $\phi(m)$ since by Equations (4) and (6),

$$\phi(m) = V(0) + 2\int_0^\infty y'PBm dt \tag{7}$$

The existence of a positive definite matrix P satisfying (6) is guaranteed by the following theorem of Lyapunov.

The equilibrium state $y = 0$ of a continuous-time, free linear stationary dynamic system

$$\frac{dy}{dt} = Ay \tag{8}$$

is asymptotically stable if, and only if, given any symmetric, positive definite matrix Q there exists a positive definite matrix P which is the unique solution of the set of $N(N+1)/2$ linear equations given by Equation (6). The function $V(t)$ given by Equation (3) is therefore a Lyapunov function for the system given by (8) since

$$V(t) = y'Py > 0; \quad y \neq 0 \qquad \dot{V}(t) = -y'Qy < 0; \quad y \neq 0 \tag{9}$$

It is our objective to show, at least formally, how the above ideas can be carried over to the design of feedback controllers for distributed systems. Some results in this direction have been obtained by Wang (2) and Pritchard (3). Wang proves that the L_2 norm is a Lyapunov function for certain hyperbolic and parabolic distributed systems. He then obtains the feedback control which minimizes the rate of change of the L_2 norm. No attempt was made, however, to choose a weighting function along with the norm in order to also satisfy some performance criteria as was done above. Pritchard shows how to use a double integral Lyapunov functional to design stable controllers for distributed parameter systems. He does not, however, show how this design method is related to Bass' technique for lumped systems. It is our belief that it is important for an engineer to appreciate the similarities as well as the differences between lumped and distributed systems in order to design effective feedback control systems.

SELECTING THE FORM OF A LYAPUNOV FUNCTION FOR DISTRIBUTED PARAMETER SYSTEMS

As our model of a distributed system, we will use various forms of the linear time invariant partial differential equation

$$\frac{\partial u}{\partial t}(x, t) = \mathcal{L}_x u(x, t) + Gf_\Omega(x, t) \tag{10}$$

for $t > 0$ and $x \in \Omega$ where Ω is a bounded, open, simply connected subset of a K-dimensional Euclidean space R_K with spatial coordinate vector $x = (x_1, \ldots, x_K)$. Let the boundary of Ω be denoted by $\partial\Omega$.

The boundary conditions can be expressed by

$$\beta u(x, t) = f_{\partial\Omega}(x, t)$$

for $t > 0$ and $x \in \partial\Omega$. The initial condition is

$$u(x, 0) = u_0(x)$$

In the above, $u = \text{col}(u_1, \ldots, u_n)$; $\mathcal{L}$ and β are linear spatial differential operators; f_Ω and $f_{\partial\Omega}$ are known functions or control vectors, and G is a matrix whose elements are continuous functions of x.

To simplify notation, we shall take x and $u(x)$ as scalars and $\Omega = (0, 1)$.

Divide Ω into N equal, disjoint sections Ω_i such that $\bigcup_{i=1}^{N} \Omega_i = \Omega$. Let y_i represent $u(x)$ for $x \in \Omega_i$, for example, $y_i = u(x_i)$ for some $x_i \in \Omega_i$. If we continue to discretize the other terms in Equation (9), we have

$$y_i \rightarrow u(x) \text{ for } x \in \Omega_i$$
$$b_i \rightarrow G(x) \text{ for } x \in \Omega_i$$
$$m_i \rightarrow f(x) \text{ for } x \in \Omega_i$$

with the exception of the sections near the boundaries where the boundary conditions (10) are incorporated in m_i where necessary.

Representing the operators $\mathcal{L}_x$ and β by some finite difference technique, the discretized system can be written as

$$\frac{dy}{dt} = Ay + Bm \tag{11}$$

$$y = \text{col}(y_1, \ldots, y_N)$$
$$m = \text{col}(m_1, \ldots, m_N)$$

Now let us take as our candidate for a Lyapunov function for (10) the norm $V(t)$ where

$$V(t) = \int_\Omega u(x, t)p(x)u(x, t)dx$$

Discretizing $u(x, t)$ as above and letting

$$p_i \rightarrow p(x) \text{ for } x \in \Omega_i$$

gives

$$V(t) = \sum_{i=1}^{N} \int_{\Omega_i} u(x, t)p(x)u(x, t)dx \cong \sum_{i=1}^{N} y_i^2 p_i/N = y'Py$$

where P = diagonal matrix with elements P_i/N.

So, the single integral functional corresponds to a diagonal matrix P in the discretized form of the Lyapunov function. But in order to make use of the theorem quoted in the introduction the only allowable restrictions on P are that it be positive definite. The additional restriction that P be diagonal prevents the application of the necessary part of the theorem. Therefore, the discretized analogy implies that the single integral functional is not sufficiently general to establish the necessary and sufficient conditions for asymptotic stability.

Let us now consider a double integral functional

$$V = \int_\Omega \int_\Omega u(x)p(x, z)u(z)dxdz \tag{12}$$

where $p(x, z) = p(z, x)$. Discretize the functional by allowing

$$p_{ij} \rightarrow p(x, z) \text{ for } x \in \Omega_i \text{ and } z \in \Omega_j$$

Then Equation (12) becomes

$$V = \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\Omega_i} \int_{\Omega_j} u(x) p(x, z) u(z) dxdz \cong y'Py \tag{13}$$

where $P = 1/N^2 \{p_{ij}\}$.

Since $p(x, z) = p(z, x)$, then $P = P'$ and we see that the double integral functional yields upon discretization a general symmetric matrix P which could be chosen to satisfy Equation (6).

The above arguments show that the discretized form of the double integral functional is sufficiently general to establish necessary and sufficient conditions for the asymptotic stability of the discretized form of a distributed system. Since the discretized approximations can usually be made arbitrarily accurate, the above development indicates that necessary and sufficient conditions for the asymptotic stability of the linear time-invariant system (10) can be established by means of the double integral functional (12). The above arguments can be extended to include systems which are multivariable and multidimensional in space.

APPLICATION OF BASS' TECHNIQUE TO DISTRIBUTED SYSTEMS

In order to apply Bass' technique to distributed parameter systems we will need the distributed parameter analog of Lyapunov's theorem for linear finite dimensional systems which was quoted in the Introduction. As yet, no such theorem exists. The following is our conjecture as to how the appropriate theorem might read. Following the conjecture we present a plausability argument in favor of it and then show how it is used in Bass' technique as extended to distributed systems.

Conjecture

The equilibrium state $u(x) = 0$ of a continuous time, free, linear stationary dynamic system

$$\frac{\partial u(x, t)}{\partial t} = \mathcal{L}_x u(x, t) \quad x \in \Omega \tag{14}$$

is asymptotically stable if, and only if, given a V-symmetric, strongly positive matrix $q(x, z)$ there exists a V-symmetric strongly positive matrix $p(x, z)$ which is the unique solution to the set of linear differential equations

$$\mathcal{L}_x^* [p(x, z)] + \mathcal{L}_z^{*\prime} [p'(x, z)] = - q(x, z) \tag{15}$$

The function $V(t)$ given by (12) is therefore a Lyapunov function for the system given by (14) since

$$V(t) = \int_\Omega \int_\Omega u'(x) p(x, z) u(z) dxdz > 0 \quad \text{for } \rho(u(\cdot)) > 0 \tag{16}$$

$$\dot{V}(t) = - \int_\Omega \int_\Omega u'(x) q(x, z) u(z) dxdz < 0 \quad \text{for } \rho(u(\cdot)) < 0 \tag{17}$$

The following are the definitions of the terms used in the conjecture:

1. A matrix kernal $K(x, z)$ is V-symmetric if for all x, z in Ω

$$K'(x, z) = K(x, z)$$

2. A matrix kernal, $K(x, z)$ is strongly positive if for every positive number ϵ there exists a positive number $\delta = \delta(\epsilon)$ such that

$$\int_\Omega \int_\Omega u'(x) K(x, z) u(z) dxdz > \delta$$

for all $u(x)$ which is square integrable in the Lebesgue sense and for which $\rho(u(\cdot)) > \epsilon$ where

$$\rho(u(\cdot)) \equiv \int_\Omega u^2(x) dx$$

This definition of strongly positive admits the possibility that $K(x, z)$ will not be composed of functions in the traditional sense but rather of distributions such as the impulse $\delta(x - z)$.

3. $\mathcal{L}_x^*[\cdot]$ is the adjoint of $\mathcal{L}_x[\cdot]$ which satisfies

$$\int_\Omega p'(x, z) \mathcal{L}_x[u(x)] dx = \int_\Omega \mathcal{L}_x^{*\prime} [p(x, z)] u(x) dx \tag{18}$$

also

$$\mathcal{L}_x'[\cdot] \equiv (\mathcal{L}_x[\cdot])'$$

When $u(x)$ and $p(x, z)$ are scalar functions then Equation (18) reduces to the standard definition of an adjoint operator. However, when $u(x)$ is a vector of functions then $\mathcal{L}_x[u(x)]$ is a vector while $\mathcal{L}_x^*[p(x, z)]$ is a matrix of functions.

In the above conjecture, the V symmetric strongly positive matrix of functions $p(x, z)$ plays the same role as does the symmetric positive definite matrix P in Equation (3) for the finite dimensional case. Similarly Equation (15) is the distributed analog of the matrix Riccati equation given by (16). The importance of the conjecture as far as the application of Bass' technique is concerned is that it guarantees the existence of a strongly positive $p(x, z)$, for any strongly positive $q(x, z)$ provided that the unforced system is inherently stable.

The previous section presented the reasoning behind the choice of the double integral of (16) as the Lyapunov function for systems given by (14). Now, by direct analogy with the finite dimensional case, the appropriate form for the rate of change of $V(t)$ must be given by (17). (Compare Equations (8) and (9) with (16) and (17).) In order to get (17) from (16) it then follows that $p(x, z)$ and $q(x, z)$ must be related through (15). To show this, we start by differentiating $V(t)$

$$\frac{dV(t)}{dt} = \int_\Omega \int_\Omega \left\{ \frac{\partial u'(x)}{\partial t} p(x, z) u(z) + u'(x) p(x, z) \frac{\partial u(z)}{\partial t} \right\} dxdz \tag{19}$$

Substituting (14) for $\partial u(x)/\partial t$ gives

$$\dot{V}(t) = \int_\Omega \int_\Omega \{\mathcal{L}_x'[u(x)] p(x, z) u(z) + u'(x) p(x, z) \mathcal{L}_z[u(z)]\} dxdz \tag{20}$$

Applying Equation (18) to Equation (20) and rearranging terms gives

$$\dot{V}(t) = \int_\Omega \int_\Omega u'(x) \{\mathcal{L}_x^*[p(x, z)] + \mathcal{L}_z^{*\prime}[p'(x, z)]\} u(z) dxdz \tag{21}$$

Equation (21) reduces to (17) if the term in parenthesis is set equal to $-q(x, z)$.

To illustrate the use of the above conjecture in Bass' design method we consider the problem of finding the appropriate forcing function $m(t)$ for the following, nonhomogeneous, scalar version of Equation (14).

$$\frac{\partial u(x,t)}{\partial t} = \mathcal{L}_x[u(x,t)] + g(x)m(t) \quad x \in \Omega \quad (22)$$

$$\beta u(x,t) = f_{\partial\Omega}(x,t) = 0$$

$$|m(t)| < 1$$

$m(t)$ is to be chosen to make Equation (22) as stable as possible and to nearly minimize the performance function $\phi(m(t))$ given by

$$\phi(m(t)) = \int_0^\infty \int_\Omega \int_\Omega u(x,t)q(x,z)u(z,t)\,dxdzdt \quad (23)$$

Following exactly the same steps as in the finite dimensional case reviewed in the introduction we introduce a Lyapunov function $V(t)$

$$V(t) = \int_\Omega \int_\Omega u(x,t)p(x,z)u(z,t)\,dxdz$$

The rate of change $V(t)$ is

$$\dot{V}(t) = \int_\Omega \int_\Omega \left[\frac{\partial u}{\partial t}(x)p(x,z)u(z) + u(x)p(x,z)\frac{\partial u(z)}{\partial t} \right] dxdz \quad (24)$$

substituting Equation (22) into (24), collecting terms and using the definition of the adjoint operator $\mathcal{L}^*$ as given by (18) yields

$$\dot{V}(t) = \int_\Omega \int_\Omega u(x)[\mathcal{L}_x^* p(x,z) + \mathcal{L}_z^* p(x,z)]\,u(z)\,dxdz \quad (25)$$

$$+ \int_\Omega \int_\Omega m(t)p(x,z)[g(x)u(z) + g(z)u(x)]\,dxdz$$

Substituting Equation (15) into the first term of (25) and using the V symmetry of $p(x, z)$ gives

$$\dot{V}(t) = -\int_\Omega \int_\Omega u(x)q(x,z)u(z)\,dxdz + 2m(t)\int_\Omega \int_\Omega p(x,z)g(z)u(x)\,dxdz \quad (26)$$

The choice of $m(t)$ which makes $V(t)$ as negative as possible subject to the constraints on $m(t)$ is

$$m(t) = -\operatorname{sign} \int_\Omega \int_\Omega p(x,z)g(z)u(x)\,dxdz \quad (27)$$

This choice of $m(t)$ also minimizes the integrand of the objective function $\phi(m(t))$ since by Equation (26) and (23)

$$\phi(m(t)) = V(0) + 2m(t)\int_0^\infty \int_\Omega \int_\Omega p(x,z)g(z)u(z)\,dxdz \quad (28)$$

The function $p(x, z)$ is found from the solution (15) with the appropriate homogeneous boundary conditions. The following example applies the above approach to the control of a system described by a simple diffusion equation with the control effort located at the boundary of the system. Another application to a distributed parameter system can be found in (4) where the technique is used to develop the control law for a binary distillation column.

AN EXAMPLE OF BASS' TECHNIQUE APPLIED TO A DISTRIBUTED SYSTEM

Problem: Find the boundary control $m(t)$ which nearly minimizes $\phi(m(t))$ where

$$\phi(m(t)) = \int_0^\infty \int_\Omega u^2(x)\,dxdt \quad (29)$$

and which makes the following system as stable as possible

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} \quad x \in \Omega = (0,1) \quad (30)$$

$$-\frac{\partial u}{\partial x}(0,t) + k\,u(0,t) = m(t) \quad \frac{\partial u}{\partial x}(1,t) = 0 \quad (31)$$

$$|m(t)| \leq 1 \quad (32)$$

The technique described in the previous section cannot be applied directly to the above problems since the boundary conditions are not homogeneous. However, it is always possible to replace a linear problem with nonhomogeneous boundary conditions with an equivalent problem with homogeneous boundary conditions by means of the impulse function.* Thus Equations (30) and (31) are equivalent to

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + m(t)\delta(x) \quad (33)$$

$$-\frac{\partial u}{\partial x}(0,t) + k\,u(0,t) = 0 \quad \frac{\partial u}{\partial x}(1,t) = 0 \quad (34)$$

The formulas developed previously can now be applied directly. To obtain a performance function with the form given by (29) from one with the form given by (23) we require that

$$q(x,z) = \delta(z - x) \quad (35)$$

The control law is obtained from (27) as (note $g(z) = \delta(z)$)

$$m(t) = -\operatorname{sign} \int_0^1 \int_0^1 p(x,z)\delta(z)u(x)\,dxdz \quad (36)$$

$$= -\operatorname{sign} \int_0^1 p(x,0)u(x)\,dx$$

Finally using Equations (15) and (25), $p(x, z)$ is obtained from

$$\frac{\partial^2 p(x,z)}{\partial x^2} + \frac{\partial^2 p(x,z)}{\partial z^2} = -\delta(z - x) \quad (37)$$

$$-\frac{\partial p}{\partial x}(0,z) + k\,p(0,z) = 0$$

$$-\frac{\partial p}{\partial z}(x,0) + k\,p(x,0) = 0$$

* It is easy to see how this is done for lumped parameter systems. The system $\dot{y} = Ay$, $y(0) = y_0$ is equivalent to $\dot{y} = Ay + y_0\,\delta(t)$, $y(0) = 0$. The equivalence between the two systems can be verified by taking the Laplace transform of each system.

$$\frac{\partial p}{\partial x}(1, z) = 0 \qquad \frac{\partial p}{\partial z}(x, 1) = 0$$

The adjoint operator $\mathcal{L}^*$ used in (37) is obtained from $\mathcal{L} = \partial^2/\partial x^2$ by integrating the left-hand side of (18) by parts twice. Alternately we could have recognized at the onset that the spatial operator in (33) is self adjoint in which case $\mathcal{L}^* = \mathcal{L}$. The solution to (37) is given by

$$p(x, z) = \begin{cases} \frac{1}{2}(x + 1/k) & 0 \leq x < z \\ \frac{1}{2}(z + 1/k) & z < x \leq 1 \end{cases} \tag{38}$$

It is shown in (5) that $p(x, z)$ as given by Equation (38) is indeed a strongly positive function. Substituting $p(x, 0) = \frac{1}{2}k$ into (36) and dropping constants yields

$$m(t) = -\,\text{sign} \int_0^1 u(x, t)dx \tag{39}$$

The feedback control law given by (39) is quite easy to implement with a finite number of measurements of $u(x, t)$ at various positions along x between zero and one. If a reasonably accurate quadrature formula is used to approximate the integral in (39), it is quite likely that ten or less measuring points will suffice. Here we see one of the advantages of having worked with the distributed system rather than some finite dimensional approximation of it. It would have been very difficult to have decided a priori how many mesh points to use to discretize (30) in order to get an accurate approximation to (39).

The control law given by Equation (39) could also have been arrived at without transforming (30) into (33). However, in order to work with (30) it is necessary to repeat each of the steps carried out in the previous section, being careful to include the fact that the boundary conditions are no longer homogeneous.

Figure 1 shows how the system given by Equation (30) responds to the feedback control law given by (39). Also shown are the responses of the same system to the feedback control policy of Wang (2) and the open loop policy of Sakawa (6). Sakawa finds the $m(t)$ which minimizes $\theta(m(t))$

$$\theta(m(t)) = \int_0^1 u^2(x, T)dx \tag{40}$$

for a fixed T and subject to Equations (30) and (31) and to the constraints that

$$0.2 \leq m(t) \leq 0.8$$

In order to be able to compare results, the feedback control law given by (39) was modified so that $m(t)$ switches between the same limits as used by Sakawa. Notice that at $t = 0.4$ both control efforts have forced $u(x, t)$ to zero. The advantage of (39) over the control policy obtained by Sakawa is that (39) can be implemented as a feedback controller with substantially less on-line computation than that required to implement the repetitive minimization of $\theta(m(t))$ [compare Equation (40)].

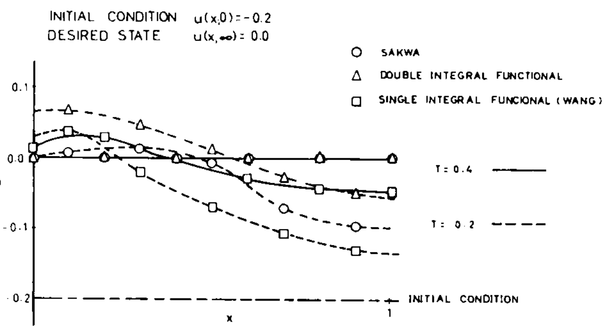

Fig. 1. Comparison of optimal and suboptimal responses for the boundary control of a diffusion system.

The feedback control law obtained by Wang (2) minimizes $\psi(m(t))$:

$$\psi(m(t)) = \frac{d}{dt} \int_0^1 u^2(x, t)dx \tag{41}$$

at each instant of time, subject to (30) and (31) and to $|m(t)| \leq 1$. The control which minimizes (41) is

$$m(t) = -\,\text{sign}\, u(0, t) \tag{42}$$

Figure 1 also shows the results of applying (42) modified so that $m(t)$ switches between -0.2 and $+0.8$. This control policy does not drive $u(x, t)$ to zero at $t = 0.4$. Thus, while the control policy given by (42) is slightly easier to implement than that given by (39), the quality of control is not as good.

CONCLUSIONS

We have shown that Bass' technique can be extended in a very natural way to treat the control of distributed parameter systems. The extension is based upon a conjecture concerning the stability of distributed systems which, even if it is not true in general, should be valid for a great majority of those systems of practical interest.

The example presented shows that Bass' technique can lead to very simple, effective, and easily implementable feedback control systems for distributed parameter processes.

NOTATION

y = process state (an n dimensional vector)
m = process inputs (an m dimensional vector)
A = process matrix ($n \times n$)
B = input matrix ($n \times m$)
ϕ = objective function
Q = weighting matrix ($n \times n$)
P = weighting matrix for Lyapunov function ($n \times n$)
$V(t)$ = Lyapunov function
k_i = bound on process input m_i
$u(x, t)$ = process state
$p(x, z)$ = weighting function for infinite dimensional Lyapunov function
$\int_\Omega$ = integral over the domain Ω
$\mathcal{L}_x$ = linear operator with respect to spatial variable x
$\mathcal{L}^*$ = adjoint of linear operator
$(\cdot)'$ = quantity $(\cdot)$ transposed

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